Ergodic Theory - Week 7

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1 Classifying measure preserving systems

P1. Show that any factor of an ergodic system is ergodic. Find an example of a non-ergodic system with an ergodic factor.

P2. Let G be a compact abelian group. Show that a group rotation by some $\alpha \in G$ is ergodic if and only if $\{n\alpha\}_{n\in\mathbb{Z}}$ is a dense subgroup of G.
Hint: You may use that any function in $L^2(G)$ has a Fourier expansion $f = \sum_\chi \widetilde{f}(\chi)\chi$ where convergence is understood to be in the $L^2(G)$-norm.

P3. We call a system $(X, \mathcal{B}, \mu, T)$ totally ergodic if for every $k \in \mathbb{N}$ the map T^k is ergodic with respect to μ.

(a) Show that if $(X, \mathcal{B}, \mu, T)$ is totally ergodic, then for any $a, b \in \mathbb{N}$ we have

$$\lim_{N\to+\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T^{an+b} f - \int f\, d\mu \right\|_{L^2(\mu)} = 0$$

(b) Show that a system is totally ergodic if and only if every eigenfunction of the system with corresponding eigenvalue that is a root of unity is constant almost everywhere.

P4. **(a)** Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system, and let $\alpha \in (0, 1)$ such that $e(\alpha)$ is an eigenvalue. Show that there exists a non-trivial group rotation that is a factor of $(X, \mathcal{B}, \mu, T)$.
Hint: When $a = r/q$ is rational (with q minimal among all such rational eigenvalues), construct a T^q-invariant set B such that $\mu(B) = 1/q$.

(b) Show that given any countable subgroup $K \leq \mathbb{S}^1$, there exists a measure preserving system $(X, \mathcal{B}, \mu, T)$ on a Borel probability space such that K is the point-spectrum of T.
Hint: To construct the system take $X = \hat{K}$, $\mu = m_X$ the normalized Haar measure on X, and T to be some appropriate group rotation.